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Classical radiation by a free spinless particle when radiation reaction force is included

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Abstract. The relativistic dynamics of a charged particle when it is interacting with its own electromagnetic field is analysed. The effect of the reaction force is relatively small but it is significant in the energy balance between the field and particle. It is found that a pulse in the radiation field is formed when the initial conditions simulate the creation of a particle.

1. Introduction

Classical theory has recently been amended, improving it to the point where the differences with quantum theory, in many cases, become marginal. The basic idea is to incorporate the uncertainty principle into classical theory, which is done by making it answer the following question: given the initial probability distribution $P_0(r)$ of finding a particle at a certain position and the probability distribution $Q_0(u)$ of having certain momentum (proper velocity or velocity for short), what is the probability distribution $P(r,t)$ at some later time t? The uncertainty principle is incorporated by assuming that $P_0(r)$ and $Q_0(u)$ are mutually related, and this relationship is taken from quantum theory. As a result, excellent agreement with quantum theory is achieved on various levels of scale. From non-relativistic scattering problems [1, 2] to the relativistic dynamics of particle in the electromagnetic (EM) field (for both scalar [3] and spin-$\frac{1}{2}$ particles [4]) and the description of spin [5] and spectroscopy [6], classical theory gives virtually the same answer as quantum theory. Of course, there are differences but they are of no fundamental importance. Based on these findings an attempt was made to reformulate one of the most difficult problems in physics, the radiation reaction force problem [7]. The basic idea behind the new formulation of this force is to note that the probability distribution $P(r,t)$ acts as the charge density and, as such, it also interacts with itself. Why should the probability distribution also act as the charge density? This is easily explained by noting that if a particle is represented by its probability distribution then the EM field produced by it (only if the particle is charged) manifests itself only through its average value. Crudely speaking, this means that, say, the scalar potential of a particle is given by the integral over the product of the probability $P(r,t)\,\mathrm{d}^3r$ of finding the particle in a small volume element around certain position r, its charge e, and the retarded Green function connecting this volume element and the observation point. However, this is exactly the field produced by $P(r,t)$ acting as the charge density. The only assumption we make is that $P(r,t)$ also interacts with itself, the idea which was used in formulating the radiation reaction force [7].

Based on this formulation of the classical radiation reaction force one can study processes where it plays a significant role. There is a whole range of them, e.g. spontaneous decay; however, we will consider one of the simplest: the radiation pattern of a free particle. Although one of the simplest problems, its solution requires, as will be shown, considerable effort. The idea is the following. The free particle, as mentioned, is represented by a probability distribution in coordinate and momentum space. These quantities define the charge and current densities at any subsequent time, and they are the source of the EM field. This field, in turn, interacts with the time evolution of the probability distribution by affecting the motion of the individual trajectory. The purpose of the paper is to analyse this self-interaction and see whether the radiation field is created as the end product.

It turns out that one of the trickiest problems is choosing the proper initial conditions for the probability distributions. Two types of initial condition will be considered. In the first, one assumes that, prior to the initial moment, the particle was bound and then set free. In the second, it is assumed that, prior to the initial moment, the probability density (and likewise the current) is zero. The latter initial condition is taken from relativistic quantum theory and describes the creation of a particle, e.g. the creation of an electron in beta decay. Radiation by electron in beta decay has been treated classically [8] by assuming that it is produced by the sudden change in the electron's velocity, from zero to a certain final value. In this paper the same problem is solved as part of a broader analysis of radiation in the dynamics of free particles. Although classical theory is used, the present approach differs in many aspects from more conventional ones, e.g. the radiation reaction force is taken into account. This is the reason why comparison between the two approaches is not entirely legitimate. In the more conventional treatment, the electron is treated as a point-like charge, with a well defined velocity vector. The radiation pattern is then analysed with respect to this vector. In the approach here radiation is produced by the probability density and probability current and the features of the radiation field reflect the properties of these sources. Thus for spherically symmetric probability distributions the radiation field is produced without the magnetic component of the EM field.

The properties of the radiation field are analysed for spin-0 particles. The 'structure' of spin-$\frac{1}{2}$ particles is more complicated, involving both the probability distribution and the circulating probability current [5]. The radiation field for this system is expected to be more complicated than for spin-0 particles and needs separate discussion. This would be a real test of ability of the classical theory to describe the properties of the radiation field produced by free particles.

2. The theory

The relativistic classical equation which describes the motion of a particle when the radiation reaction force is included can be easily written down once the model, described in section 1, is assumed. The probability distribution $P(\boldsymbol{r}, x_0)$ and, likewise, the probability current $\boldsymbol{j}(\boldsymbol{r}, x_0)$ are represented by a bunch of trajectories the initial conditions of which are taken randomly from the initial distributions in the coordinate and momentum of the particle. At each instant in time the probability density and probability current produce the EM field, and this field interacts through the Lorentz force with each trajectory. This self-interacting set of equations can be written as [7]

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}s} = \frac{1}{m\kappa}\boldsymbol{K} + \alpha \boldsymbol{E}_{\text{react}} \tag{1}$$

where $\boldsymbol{K}$ is some external force acting on the particle. For a free particle it is zero; this case will be assumed in further discussion. m is the mass of the particle, κ is its appropriate

Compton's wavenumber ($\kappa = mc/\hbar$), and α is the fine-structure constant. Equation (1) is obtained after the coordinates are scaled with respect to κ, so when we write $\boldsymbol{r}$ this means $\kappa\boldsymbol{r}$, and s means $s\kappa$ (s is the proper time). In these scaled coordinates, and when the charge e is taken out as the factor from the four-potential, the radiation reaction force is

$$\boldsymbol{F}_{\text{react}} = \boldsymbol{E}\sqrt{1+u^2} + \boldsymbol{u}\times\boldsymbol{B} \tag{2}$$

where $\boldsymbol{u} = \mathrm{d}\boldsymbol{r}/\mathrm{d}s$ and

$$\begin{aligned}\boldsymbol{E} &= \int \mathrm{d}^3r_0\,\mathrm{d}^3u_0\,\rho(\boldsymbol{r}_0,\boldsymbol{u}_0)\boldsymbol{\epsilon}(x_0^{\text{ret}})\\ \boldsymbol{B} &= \int \mathrm{d}^3r_0\,\mathrm{d}^3u_0\,\rho(\boldsymbol{r}_0,\boldsymbol{u}_0)(\boldsymbol{R}/R)\times\boldsymbol{\epsilon}(x_0^{\text{ret}}).\end{aligned} \tag{3}$$

Both $\boldsymbol{E}$ and $\boldsymbol{B}$ in these units are dimensionless. The field $\boldsymbol{\epsilon}$ is

$$\boldsymbol{\epsilon}(x_0^{\text{ret}}) = \frac{R^3}{(\boldsymbol{R}\boldsymbol{A})^3}[\boldsymbol{A} + \boldsymbol{R}\times\{\boldsymbol{A}\times[\dot{\boldsymbol{u}}(1+u^2) - \boldsymbol{u}(\boldsymbol{u}\dot{\boldsymbol{u}})]\}] \tag{4}$$

where

$$\boldsymbol{A} = \boldsymbol{R}\sqrt{1+u^2} - R\boldsymbol{u}. \tag{5}$$

The dot designates the derivative with respect to s. The coordinate $\boldsymbol{R}$ is defined by $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}_{r_0u_0}(x_0^{\text{ret}})$, where $\boldsymbol{r}_{r_0u_0}(x_0^{\text{ret}})$ means the coordinate of the particle at the retarded time if its initial conditions were $\boldsymbol{r}_0$ and $\boldsymbol{u}_0$. The retarded time is defined by the equation

$$x_0^{\text{ret}} = x_0 - |\boldsymbol{r} - \boldsymbol{r}_{r_0u_0}(x_0^{\text{ret}})| \tag{6}$$

where x_0 stands for $\kappa x_0 = \kappa ct$. The velocity $\boldsymbol{u}$ and its derivative $\dot{\boldsymbol{u}}$ in (4) are evaluated at $x_0 = x_0^{\text{ret}}$.

$\rho_0(\boldsymbol{r}_0,\boldsymbol{u}_0)$ is the initial probability distribution in the phase space and it is given as a product $P_0(\boldsymbol{r}_0)Q_0(\boldsymbol{u}_0)$, where $P_0(\boldsymbol{r}_0)$ is the initial probability density of finding a particle at the position $\boldsymbol{r}_0$ and $Q_0(\boldsymbol{u}_0)$ is the initial probability density of finding a particle with the velocity $\boldsymbol{u}_0$. For any later time x_0, the phase space distribution $\rho(\boldsymbol{r},\boldsymbol{u};x_0)$ is not product separable. In our analysis one of the initial distributions will be taken to be arbitrary but the others will be determined from quantum theory. In the non-relativistic theory the relationship between the two is easily obtained; however, in relativistic theory this relationship is not straightforward. First, there is no unique generalization of the non-relativistic quantum theory in the relativistic domain. The choice is between the Klein–Gordon (KG) or the Dirac equations. For the KG equation, which will be assumed in this paper, the probability density is defined as

$$P(\boldsymbol{r},x_0) = -\operatorname{Im}\left[\psi^*\frac{\partial\psi}{\partial x_0}\right]. \tag{7}$$

A brief note concerning this quantity is appropriate. In the usual interpretation (7) is defined as the charge density; however, we will regard it as the probability density.

If we write the following equation for the time evolution of the wavefunction

$$\psi = \int \mathrm{d}^3u\,A(\boldsymbol{u})\mathrm{e}^{-\mathrm{i}\epsilon_u x_0 + \mathrm{i}\boldsymbol{r}\boldsymbol{u}} \tag{8}$$

where only the positive energy solutions are considered, then at $x_0 = 0$ the distribution is [9]

$$P_0(\boldsymbol{r}) = \tfrac{1}{2} \int \mathrm{d}^3u\, \mathrm{d}^3u'\, (e_u + e_{u'}) A(\boldsymbol{u}) A(\boldsymbol{u}') \mathrm{e}^{\mathrm{i}\boldsymbol{r}(\boldsymbol{u}-\boldsymbol{u}')} \tag{9}$$

where $e_u = \sqrt{1+u^2}$ and $A(\boldsymbol{u})$ is the amplitude which is obtained from the initial probability distribution. In general $A(\boldsymbol{u})$ is difficult to find, given $P_0(\boldsymbol{r})$ and the initial probability current $\boldsymbol{J}_0(\boldsymbol{r})$, so we will adopt a trial and error procedure. As it turns out, if we want $P_0(\boldsymbol{r})$ to represent the Gaussian distribution, of the form

$$P_0(\boldsymbol{r}) = \mathrm{e}^{-r^2/a^2} \tag{10}$$

then

$$A(\boldsymbol{u}) = \mathrm{e}^{-a^2u^2/2}/\sqrt{1+u^2} \tag{11}$$

which differs from the non-relativistic relationship in the factor $(1+u^2)^{-1/2}$. Given this relationship $Q_0(\boldsymbol{u})$ is

$$Q_0(\boldsymbol{u}) = N\mathrm{e}^{-a^2u^2}/\sqrt{1+u^2} \tag{12}$$

where the normalization factor N is obtained from

$$\int \mathrm{d}^3r\, P_0(\boldsymbol{r}) = 1 = (2\pi)^3 \int \mathrm{d}^3u\, e_u A^2(u). \tag{13}$$

Solving equation (1) is very difficult, one of the major difficulties being the calculation of the retarded time x_0^{ret}. If the effect of the retarded time is neglected the problems are still great but manageable; however, this approximation applies only to the non-relativistic case.

Equation (1) is, in fact, a set of equations, where the coupling comes from the integrals (3), since they require knowledge of all trajectories (through the probability density and the probability current), albeit, at some earlier time x_0^{ret}. Therefore, the way to solve (1) is to choose a set of random initial conditions from the distributions $P_0(\boldsymbol{r})$ and $Q_0(\boldsymbol{u})$, and start integrating the appropriate equations of motion. If N trajectories were chosen then this would mean $6N$ differential equations need to be solved. At each instant of time the probability density and current are calculated, from which the EM field is obtained. The EM field, in turn, interacts through the Lorentz force with each trajectory, with a certain delay, due to its finite speed of travel.

The major obstacle, as we have already mentioned, is how to deal with the retarded time efficiently. One way is described here but it may not be the only possible one. We start from the four-potential written in its most primitive version

$$A^\mu = 2e \int_{-\infty}^{x_0} \mathrm{d}x_0' \int \mathrm{d}^3r'\, \mathrm{d}^3u\, \delta[(x_0 - x_0')^2 - (\boldsymbol{r}-\boldsymbol{r}')^2] \frac{u^\mu}{\sqrt{1+u^2}} \rho(\boldsymbol{r}', \boldsymbol{u}, x_0') \tag{14}$$

and, if the delta function is replaced by its Fourier transform, we obtain

$$A^\mu = \frac{e}{\pi} \int_{-\infty}^{x_0} \mathrm{d}x_0' \int_{-\infty}^{\infty} \mathrm{d}w\, \mathrm{e}^{\mathrm{i}w(x_0-x_0')^2} \int \mathrm{d}^3r'\, \mathrm{d}^3u\, \frac{u^\mu}{\sqrt{1+u^2}} \rho(\boldsymbol{r}', \boldsymbol{u}, x_0') \mathrm{e}^{-\mathrm{i}w(\boldsymbol{r}-\boldsymbol{r}')^2}. \tag{15}$$

The analysis is greatly simplified for a spherically symmetric distribution $\rho(r, u, x_0)$. The probability density $P(r, x_0)$ is then spherically symmetric and the current $j(r, x_0)$ has only a radial component. The $\mu = 0$ component of the potential A^μ (the scalar potential) is then

$$A^0(r, x_0) = \frac{e}{\pi} \int_{-\infty}^{x_0} dx_0' \int_{-\infty}^{\infty} dw\, e^{iw(x_0 - x_0')^2} \int d^3r'\, P(r', x_0') e^{-iw(r-r')^2} \tag{16}$$

(we have used the identity $u^0 = \sqrt{1+u^2}$) and the integration over the angles of r' can be done immediately, with the result

$$A^0(r, x_0) = \frac{4e}{r} \int_{-\infty}^{x_0} dx_0' \int_0^{\infty} dr'\, r' P(r', x_0') \int_0^{\infty} dw\, \cos[(x_0 - x_0')^2 - r^2 - r'^2] \frac{\sin(2wrr')}{w}. \tag{17}$$

The integral over w is non-zero and equal to $\pi/2$ if [10]

$$2rr' > |(x_0 - x_0')^2 - r^2 - r'^2| \tag{18}$$

which gives the limits on x_0'

$$x_0 - r - r' < x_0' < x_0 - |r - r'|. \tag{19}$$

Furthermore, by definition

$$P(r, x_0) = \int d^3u\, \rho(r, u, x_0) \tag{20}$$

and if we define the radial distribution by

$$p(r, x_0) = 4\pi P(r, x_0) r^2 \tag{21}$$

then the scalar potential is finally

$$A^0(r, x_0) = \frac{e}{2r} \left[\int_0^r \frac{dr'}{r'} \int_{x_0 - r - r'}^{x_0 - r + r'} dx_0'\, p(r', x_0') + \int_r^{\infty} \frac{dr'}{r'} \int_{x_0 - r - r'}^{x_0 + r - r'} dx_0'\, p(r', x_0') \right]. \tag{22}$$

Similarly we derive the vector components of A^μ (the vector potential). It is given by

$$A(r, x_0) = \frac{e}{\pi} \int_{-\infty}^{x_0} dx_0' \int_{-\infty}^{\infty} dw\, e^{iw(x_0 - x_0')^2} \int d^3r'\, J(r', x_0') e^{-iw(r-r')^2} \tag{23}$$

where J has only a radial component which can be written as

$$J = J\hat{r} \tag{24}$$

where $\hat{r}$ is the unit vector. The integral over the angles of r' are now easily evaluated, and give for the vector potential

$$A = \frac{4e}{r} \hat{r} \int_{-\infty}^{x_0} dx_0' \int_0^{\infty} dr'\, r' J(r', x_0') \int_0^{\infty} dw\, \frac{\sin[(x_0 - x_0')^2 - r^2 - r'^2]}{w} \times \left[\cos(2wrr') - \frac{\sin(2wrr')}{2wrr'} \right]. \tag{25}$$

The range of x_0' for which the integral over w is non-zero is similar as that in the derivation of the scalar potential. The detailed discussion is omitted, and the final result for $\boldsymbol{A}$ is

$$\boldsymbol{A} = -\frac{e}{4r^2}\hat{r}\left\{\int_0^r \frac{\mathrm{d}r'}{r'^2}\int_{x_0-r-r'}^{x_0-r+r'} \mathrm{d}x_0'\, j(r',x_0')f(x_0,x_0',r,r') + \int_r^\infty \frac{\mathrm{d}r'}{r'^2}\int_{x_0-r-r'}^{x_0+r-r'} \mathrm{d}x_0'\, j(r',x_0')f(x_0,x_0',r,r')\right\} \tag{26}$$

where

$$f(x_0,x_0',r,r') = (x_0-x_0')^2 - r^2 - r'^2. \tag{27}$$

We used the definition of the radial current density, $j = 4\pi J r^2$.

From the explicit form of the four-potential we can derive the Lorentz force. The magnetic component of the EM field is zero because $\boldsymbol{A}$ has only a radial component, whilst the electric component is given by

$$\boldsymbol{E} = -\nabla A^0 - \partial \boldsymbol{A}/\partial x_0 \tag{28}$$

or more explicitly

$$\begin{aligned}\boldsymbol{E} = {}& \frac{e\hat{r}}{2r^2}\int_0^r \frac{\mathrm{d}r'}{r'^2}\int_{x_0-r-r'}^{x_0-r+r'} \mathrm{d}x_0'\, g(x_0,x_0',r,r') + \frac{e\hat{r}}{2r^2}\int_r^\infty \frac{\mathrm{d}r'}{r'^2}\int_{x_0-r-r'}^{x_0+r-r'} \mathrm{d}x_0'\, g(x_0,x_0',r,r') \\ & - \frac{e\hat{r}}{2r}\int_0^\infty \frac{\mathrm{d}r'}{r'}[p(r',x_0-r-r') + j(r',x_0-r-r')] \\ & - \frac{e\hat{r}}{2r}\int_0^r \frac{\mathrm{d}r'}{r'}[-p(r',x_0-r+r') + j(r',x_0-r+r')] \\ & - \frac{e\hat{r}}{2r}\int_r^\infty \frac{\mathrm{d}r'}{r'}[p(r',x_0+r-r') + j(r',x_0+r-r')]\end{aligned} \tag{29}$$

where

$$g(x_0,x_0',r,r') = r'p(r',x_0') + (x_0-x_0')j(r',x_0'). \tag{30}$$

It can be easily verified that, in the limit $r \to 0$, the electric component $\boldsymbol{E}$ goes to zero, as it should by the consideration of symmetry.

3. The initial conditions

One of the most difficult problems which needs to be addressed now is: how do we decide on the initial conditions, i.e. about $P_0(\boldsymbol{r})$ and $Q_0(\boldsymbol{u})$? Without the radiation reaction force included the answer is simple: any reasonable choice of $P_0(\boldsymbol{r})$ at, say, $x_0 = 0$ is valid and $Q_0(\boldsymbol{u})$ is determined as discussed in section 2. The problem arises when the radiation reaction force is included because there is, even in principle, no way of controlling it, i.e. being able to turn it on in the prescribed manner or at the prescribed time. As a consequence the initial condition at some fixed time x_0 has no meaning since A^μ depends on the distribution and the current prior to that moment, which means that the time evolution of the probability distribution also depends on its values prior to that moment. Therefore, it is not enough to know the probability distribution at some initial moment x_0, we also need to know it at all times before that.

A possible choice for the initial conditions is to assume that the particle is bound until the initial instant $x_0 = 0$, and then it is set free (we will refer to this as a type I initial condition). In such a case, prior to that instant the probability distribution is stationary and the probability current is zero, which is always satisfied if the particle was in the ground state, even when the radiation reaction force is included. The excited states, in general, are not stationary when the radiation reaction force is included, the resulting effect being spontaneous decay. It is obvious that this initial condition depends on the choice of interaction which confines the particle but for modelling purpose one can assume a certain distribution $P_0(r)$ and not worry about the potential which, together with the radiation reaction force, produces it. In this paper we will discuss an initial condition of type (10), whilst in the momentum space it is given by (12). We do not discuss the form of potential which produces this stationary probability distribution.

There are other ways of choosing initial conditions but one should make sure they are physically reasonable. For example, in non-relativistic quantum theory, and similarly in its relativistic generalization, the initial condition is different from the one previously mentioned. For the sake of completeness we will discuss it briefly using the example of a free particle, with and without the radiation reaction force included. We will refer to it as a type II initial condition.

In quantum theory the time evolution of the wavefunction is conveniently represented in the form [11]

$$\psi(r, x_0) = \int \mathrm{d}^3 r' \, G(r - r', x_0) \psi(r', 0) \tag{31}$$

where the initial condition is explicitly represented by the function in the integral. This form is quite general and differs only in the details when one seeks a solution of the KG or the Dirac equation (or the non-relativistic Schrödinger equation). The Green function, or the propagator $G(r - r', x_0 - x_0')$, is the solution of the relativistic quantum equation. For example, in the case of the KG equation, when no radiation reaction is included, this equation is of the form (in the appropriate units)

$$\left[-\Delta^2 + \frac{\partial^2}{\partial x_0^2} + m^2\right] G(r - r', x_0 - x_0') = -\delta(r - r')\delta(x_0 - x_0'). \tag{32}$$

The initial wavefunction $\psi(r, 0)$ can be a mixture of positive and negative frequency states and the choice of the Green function is such that for $x_0 > 0$ only the positive ones are propagated, whilst for $x_0 < 0$ only the negative ones [11]. Therefore, if the initial state is chosen with only the positive frequency states (this is always possible, as shown on various occasions), then the wavefunction (31) for $x_0 < 0$ is zero, the same as in non-relativistic quantum theory. This is not a problem if the history of the wavefunction is immaterial as in the case when no radiation reaction is included. However, if the history is relevant then the time evolution of the wavefunction with this initial condition suffers from serious drawbacks, e.g. the law of conservation of the probability is not satisfied. We will come back to this point shortly.

As mentioned the problems arise when the radiation reaction force is included. The relativistic quantum equation for the particle is now coupled with the equations for the EM field (the four-potential), and the set, when the KG type of equation is considered, is (in the appropriate units)

$$\begin{aligned}&\left[-(\nabla+\mathrm{i}eA)^2+\left(\frac{\partial}{\partial x_0}+\mathrm{i}eA^0\right)^2+m^2\right]\psi=0\\&-\Delta A^\mu+\frac{\partial^2}{\partial x_0^2}A^\mu=j^\mu\end{aligned}\tag{33}$$

where the four-current j^μ for the scalar particle is

$$j^\mu=-\mathrm{Im}\left[\psi^*\left(\frac{\partial}{\partial x_\mu}+\mathrm{i}eA^\mu\right)\psi\right].\tag{34}$$

It should be noted that if we go over to QED the coupling of the EM field (i.e. photons) *to the particle is represented by the same type of equation as (33)* (with the KG equation replaced by Dirac's) except that it is the set of equations for the quantized fields [12].

We are now confronted with a problem of how to solve the set (33) and, in particular, how to choose the initial conditions. One point is certain: the set is no longer linear in the wavefunction ψ and, therefore, many of the techniques for solving linear equations, in particular the perturbation method, should be used with the utmost caution [7]. However, we are not going to discuss this further but move on to analyse the problem of the initial conditions.

If the solution of (33) is known, and ψ is written in the form (31), then the four-potential A^μ is calculated as discussed in the previous section. However, the initial conditions which we now use reflect the properties of the Green function. For example, if solution (31) involves only the positive frequency states then the four-potential A^μ is zero for $x_0<0$. The initial condition appears to be unrealistic, as argued before; however, it approximately represents the creation of a particle, e.g. the electron in beta decay. In this process a particle of the opposite charge is created (proton) but that would involve discussion of multi-particle systems, which we are not currently considering. Later we will discuss solutions based on this initial condition; however, we will neglect the fact that a particle of opposite charge is created.

4. Numerical method for solving the equations of motion

The basic equations of motion which we need to solve in our analysis are given by (1), where, for a free particle, the external force $\boldsymbol{K}$ is zero. The set (1) is still not in final form, because it should be given in the real time x_0 rather than the proper time s. If we use the transformation

$$\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{u}=\frac{\dot{\boldsymbol{v}}+\boldsymbol{v}\times(\boldsymbol{v}\dot{\boldsymbol{v}})}{(1-v^2)^2}\tag{35}$$

where the dot now designates the derivative with respect to the real time x_0, and $\boldsymbol{v}$ is the velocity in the real time, it can be shown that

$$\ddot{\boldsymbol{r}}=\alpha(1-v^2)[\boldsymbol{F}_{\mathrm{react}}-\boldsymbol{v}(\boldsymbol{v}\boldsymbol{F}_{\mathrm{react}})]\tag{36}$$

replaces the set (1) but now in the real time x_0. The reaction force is given by (2), where the magnetic component is zero and the electric component is given by (29). The set of equations (36) is very complicated, and it is a set because on the right-hand side the force is given in terms of the probability distribution and the probability current, which can only be calculated if all the trajectories are calculated at the same time. In this way we are able to calculate the Lorentz force by first sampling all the trajectories in order to produce the distribution and the current, and then calculate the field from which the force on particular trajectory is calculated. However, all the trajectories cannot be taken into account (there are an infinite number of them) and therefore we must work with a finite set of them, taken randomly from the initial probability distributions $P_0(r)$ and $Q_0(u)$. This will inevitably introduce errors in the calculations, in the form of fluctuations of random numbers. The fluctuations do not represent a problem, and one can always improve on this point if larger numbers of trajectories are taken into account. However, one particularly unpleasant problem arises which needs to be mentioned. Very often in the calculations there is cancellation of large numbers in order to produce a result which is small. This is always an unpleasant numerical problem but particularly so when random numbers are involved. Namely, fluctuations of random numbers do not decrease rapidly with the increasing number of trajectories, in fact they go to zero as the inverse of the square root of the number of the trajectories. Therefore if this problem is encountered, as it is in our analysis, particular care should be taken that the influence of these numerical instabilities is made as small as possible. In several examples we will obtain an EM field with large fluctuations when they should be zero. This is exactly the problem which we mentioned; however, the influence of the results on the time evolution is negligible. Overall, where the field is substantial, numerical instabilities of this kind are small.

Fluctuations in random numbers have the greatest effect on the probability distribution and current. However, the field, because the integrals in (29) smooth out these fluctuations, is relatively well behaved and therefore interpolations are possible. This is particularly important because the Lorentz force is not simply related to the probability density and current. In fact, we must keep track of all the probability distributions and the probability currents from the past, and when the integrals in (29) are calculated it is necessary to take all of them into account. In practice the time steps were conveniently chosen, and at each one of them both these quantities were stored at certain space intervals. The integrals were calculated by the trapezoidal rule, but were required when interpolation was necessary. This was particularly the case when the solution was propagated between two time steps (the numerical algorithm for solving sets of differential equations, developed by Shampine and Gordon [13], was used). Various checks were made in order to ensure stability of the solutions, and the examples which we present were all done with these checks.

Summarizing: the procedure for solving set (36) is as follows. At time $x_0 = 0$ a set of $6N$ randomly chosen initial conditions is taken for the radial coordinate and the radial proper velocity (from the distributions discussed previously) and the appropriate spherical angles (uniformly distributed within their domain of definition) of these two parameters. The time step is fixed and propagation within this time step started. At each instant the radial probability distribution and the radial current are calculated using the usual sampling method [1]. From these results, and those of previous time steps, the integrals in (29) are calculated for the spatial points where the particular trajectory is. At this point interpolation is necessary, which was done by splines. In this way the force on the right-hand side of (36) was calculated, and hence propagation of the solution achieved.

5. Examples

Various examples will be discussed in this section, covering the most typical cases. Two limiting ones are of interest, and they are distinguished according to the width of the initial probability distributions (it should be recalled that the coordinates are scaled with respect to the Compton wavelength). Those with widths greater than 1 ($a \gg 1$ in (10), which we use in this paper) fall into the non-relativistic domain and the ones with widths much less than 1 ($a \ll 1$ in (10)) fall into the relativistic domain. Their time evolution is entirely different, as can be seen in figure 1, where the initial probability distribution $p_0(r)$ (full curve) for two values of a is shown after a certain time interval (step-like line). In their dynamics no radiation reaction force is included. One typical feature of the probability distributions which was mentioned in the previous section should be noted: the distributions are not smooth, instead they are given by a set of points which fluctuate around some average value. This arises as a result of using a finite number of randomly chosen initial conditions. For each value of the width we show two types of distribution: the three-dimensional distribution in coordinates $P(r,t)$ and the distribution in the radial coordinate $p(r,t) = r^2 P(r,t)$. The fluctuations of $P(r,t)$ for small r are much larger than those of $p(r,t)$, and therefore it is more convenient to work in spherical coordinates. From now on when we talk about the probability distribution in the coordinates we mean the distribution of spherical angles and the radial coordinate.

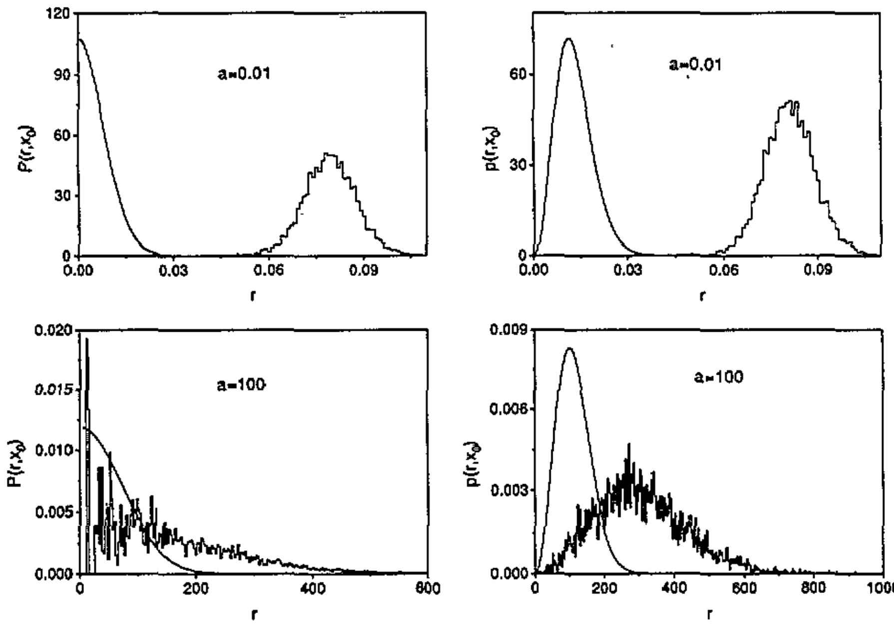

Figure 1. Typical time evolution of narrow ($a = 0.01$) and broad ($a = 100$) probability distributions without the radiation reaction force included. The distributions at $x_0 = 0$ are shown by a full curve and, after a certain time, they are shown by a step-like line. Two types of probability distribution are shown: the three-dimensional $P(r, x_0)$ and the radial $p(r, x_0) = r^2 P(r, x_0)$.

Figure 1 shows two entirely different time behaviours of the probability distributions, depending on the choice of a. In the non-relativistic limit ($a = 100$) the distribution only spreads, i.e. its width increases in time. On the other hand, in the relativistic limit ($a = 0.01$) a radial probability 'pulse' is produced which travels at nearly the speed of light, and it is relatively stable. Because of this 'pulse' one can associate with it an average momentum and energy, the latter being given by

$$E_{\text{particle}} = \int \mathrm{d}^3 r \, \mathrm{d}^3 u \, \rho(\boldsymbol{r}, \boldsymbol{u}, x_0)\sqrt{1+u^2}. \tag{37}$$

The factor mc^2 is omitted in the definition of E_{particle}.

When the radiation reaction force is included in the dynamics of the probability distribution we expect to notice a change in its properties and in the pattern of the electromagnetic field. We will consider two different initial conditions for the probability distributions: types I and II described in section 3.

When the type I initial condition is assumed the dynamics of the probability distributions in the non-relativistic case ($a \gg 1$) is expected to be simple. The basis for this conclusion is straightforward. The spread of velocities in the initial distribution is non-relativistic, and hence the effect of retardation is negligible. Indeed this is the case and, in the first instant, this effect can be neglected. However, the dynamics of the probability distribution is far from simple. This limit was discussed quite extensively [7] and it was found that on a long time scale the initial probability distribution does not stay centred around the origin, but a 'pulse' is produced, as shown in figure 2 for an extreme case when $a = 10^5$. After $x_0 = 10^{10}$ units of time the initial distribution (full curve) has moved away from the origin (step-like line), with a well defined front which is moving with a certain velocity. The consequences of this finding are far reaching [7], but for our discussion they are not of immediate importance. Far more interesting, from the point of the EM field, is the relativistic case, i.e. the dynamics of the narrow initial probability distributions.

A typical example of the time evolution of a narrow probability distribution is shown in figure 3. The broken curve is the distribution itself and its scale is in convenient units so that it can be related to the electric field E_r (full curve), which has only the radial component. The electric field is in dimensionless units, as defined in section 2. The width of the initial distribution is $a = 0.01$, and the times at which the sequence of figures is made is indicated in figure 3.

Three characteristic features of the electric field are noted. For large r, outside the probability distribution, the electric field is the ordinary Coulomb-type field of a point-like charge e. Its functional dependence on r is r^{-2}. Inside the distribution the field gradually goes to zero and, in the region of small r, again outside the distribution, it is zero or it oscillates around zero. These oscillations are artificial, and they are the result of numerical instabilities in the calculation with random numbers. The field for small r is the result of cancellation of two components in (28), one from the vector potential and the other from the scalar potential. Each component is large and, therefore, when they cancel each other out numerical instabilities are expected, which is serious when random numbers are involved. In the previous section we discussed this point in detail and, in order to test that this is indeed the case, the calculation was repeated for a smaller number of initial conditions. The results in figure 3 were obtained with 5000 sets of initial conditions (30 000 coupled equations), and in figure 4 a comparison with the probability distribution and the electric field when 1000 sets (broken line) of initial conditions is chosen is made. The results are nearly identical, save for the random number fluctuations. However, oscillations in the electric field for small r are much more enhanced for a smaller number of the initial conditions. It is, therefore, expected that as the number of the initial conditions is increased the electric field would indeed go to zero.

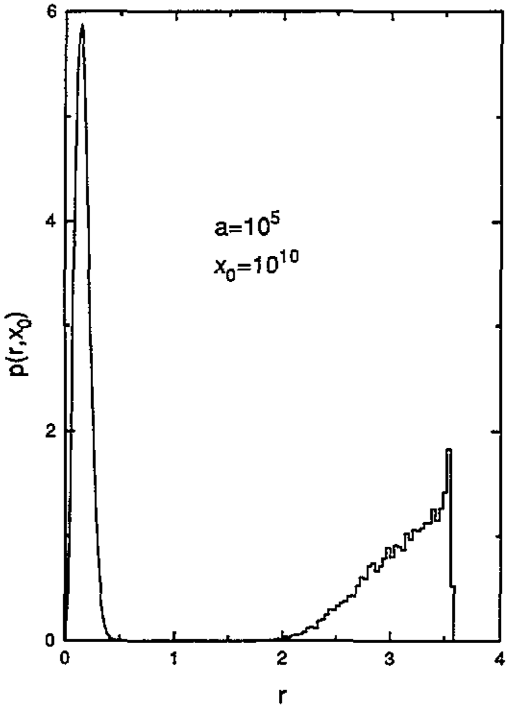

Figure 2. Time evolution (step-like line) of a very broad initial probability distribution (full curve) when the radiation reaction force is included. The initial condition is of type I.

Very narrow probability distributions disintegrate quickly, producing the radial 'pulse' which travels at nearly the speed of light. The electric field follows the motion of the distribution and eventually disappears, as shown in figure 3. The energy of the field is, however, converted into the energy of the particle so that the total energy of the system is constant. The energy of the particle has already been defined in (37) (it is given in the units of mc^2). The energy of the field, on the other hand, is (in the same units)

$$E_{\text{field}} = \frac{\alpha}{8\pi} \int \mathrm{d}^3 r\, E^2 \tag{38}$$

where we have used parametrization of the field defined in section 2. The total energy of the system is therefore

$$E_{\text{tot}} = E_{\text{field}} + E_{\text{particle}} \tag{39}$$

and it is conserved, which can be shown by proving

$$\frac{\partial}{\partial x_0} E_{\text{tot}} = 0. \tag{40}$$

This is indeed true provided $\rho(r, u, s)$ satisfies the relativistic Liouville equation

$$\frac{\partial \rho}{\partial x_0} = -v\nabla_r \rho - F\nabla_u \rho \tag{41}$$

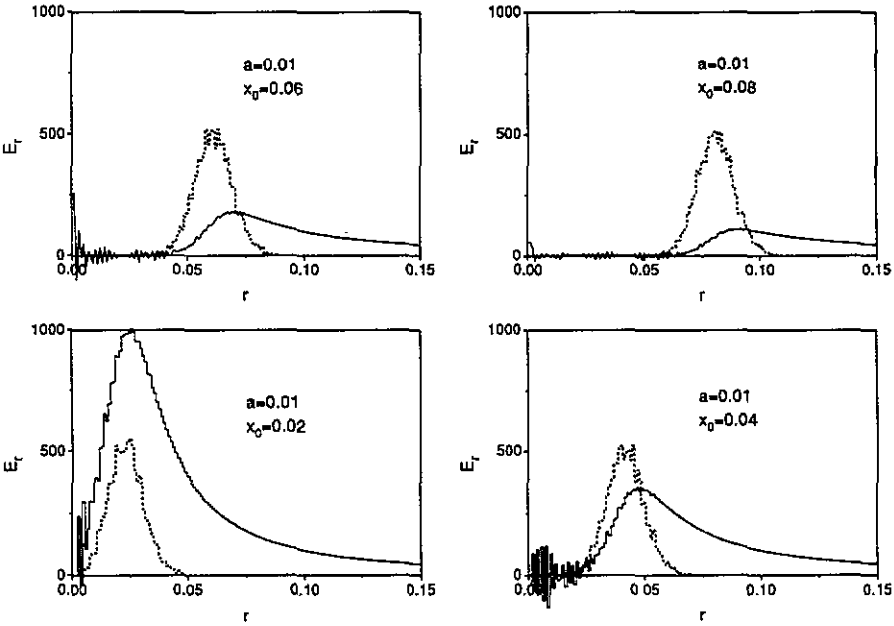

Figure 3. Time evolution of the electric field E_r for a narrow initial probability distribution, with type I initial conditions. The probability distribution (broken curve and not to scale) is shown for comparison. Oscillations in the field are due to numerical instabilities.

where the indices of the nabla operator mean gradients with respect to the r and u variables, respectively. The force F is given by (1), and in our case $F = \alpha F_{\text{react}}$.

Figure 5 shows the calculation of the energy of the particle and the field. The energy of the field goes to zero, or at least it appears so. Likewise the energy of the particle increases, however, by only a small fraction of its entire energy. The effect of the field on the dynamics of the particle is, therefore, relatively small. The total energy E_{tot} of the system is also shown (broken curve), and it is constant. The small deviation of E_{tot} from constancy for small x_0 is the result of numerical instabilities.

A calculation with a narrower initial probability distribution ($a = 0.001$) was done and the results are also shown in figure 5. The effect of the field on the energy of the particle is larger in absolute magnitude, compared with the broader initial distribution ($a = 0.01$); however, in relative magnitude the effect is nearly the same.

The initial condition which we analysed does not produce the EM field which has the property of a radiation field. In other words, the maximum of the field goes to zero faster than r^{-1} as time progresses and, hence, the total field energy goes to zero. Entirely different properties for the EM field are obtained for the type II initial conditions, i.e. when the probability distribution is zero for $x_0 < 0$. This condition implies that, for $x_0 < 0$, the field is also zero and, at $x_0 = 0$, it is created near the origin, in the vicinity of $p_0(r)$. The field will tend to fill up the space, and this disturbance propagates at the speed of light. However, there is always a region of space, beyond approximately $r > x_0$, where the field is zero. This qualitative description of the field is confirmed in the calculations. For a narrow initial probability distribution ($a = 0.01$) the results for the electric field are shown in figure 6 (full curve), where the probability distribution is also shown (broken curve) for comparison.

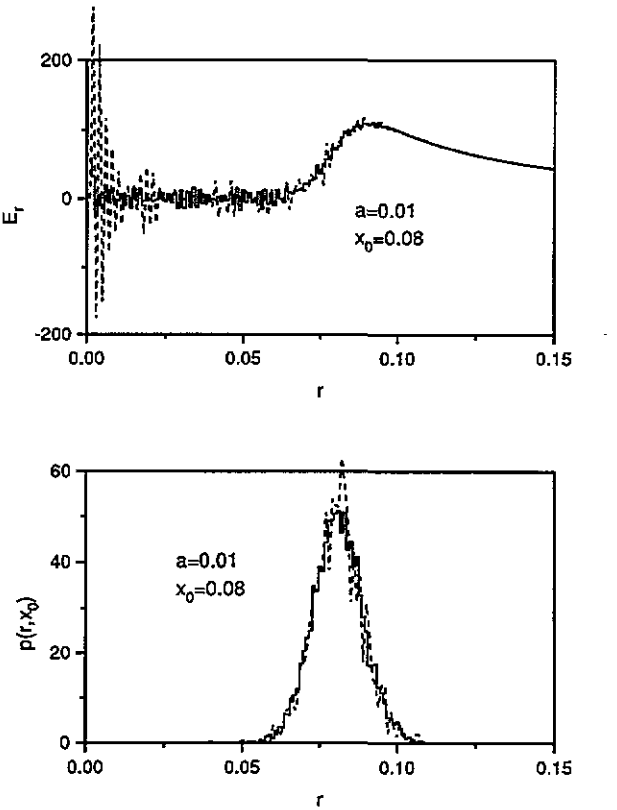

Figure 4. Probability distribution $p(r, x_0)$ and the electric field E_r as a function of the number of random initial conditions for trajectories. Increasing the sets of initial conditions from 1000 sets (broken curve) to 5000 (broken curve) reduces the amplitude of the oscillations in the electric field for small r.

The electric field starts from zero, acquires a large amplitude in the vicinity of the origin and then travels in the form of a pulse with its amplitude slowly decaying with time. This feature of the field is a result of the dynamic properties of the probability distribution which also travels at nearly the speed of light, in the form of a 'pulse' going radially. Since the velocity of the 'pulses' is nearly the same the entire field will be confined within the probability distribution and, hence, their mutual interaction will be strong. Eventually, part of the field will overtake the probability distribution but that may take a very long time when the latter is very narrow. In particular we note that no Coulomb tail was formed, at least within the time interval in figure 6, which is expected to happen.

Calculation of the energy of the field and the particle reveals interesting details. Figure 7 shows that the energy of the field rises from zero, as it should, and then acquires a constant value. The implication is that E_r has the properties of the radiation field, because the energy of the field can only be constant if its maximal amplitude does not go to zero faster than r^{-1}. Therefore, in the process of creating a particle, which the assumed initial condition describes, radiation is produced but not of the type to which we are accustomed. This radiation field does not have a magnetic component.

On the other hand, the energy of the particle is constantly rising, and so is the total

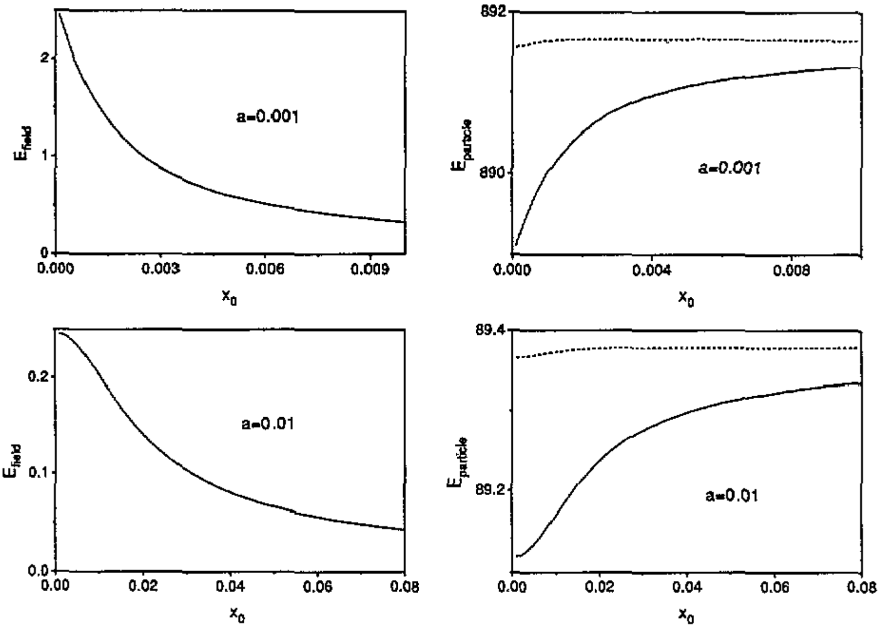

Figure 5. Time dependence of the energy of the field E_{field} and the particle $E_{particle}$ for two narrow initial probability distributions of type I. The broken curve shows the total energy of the system

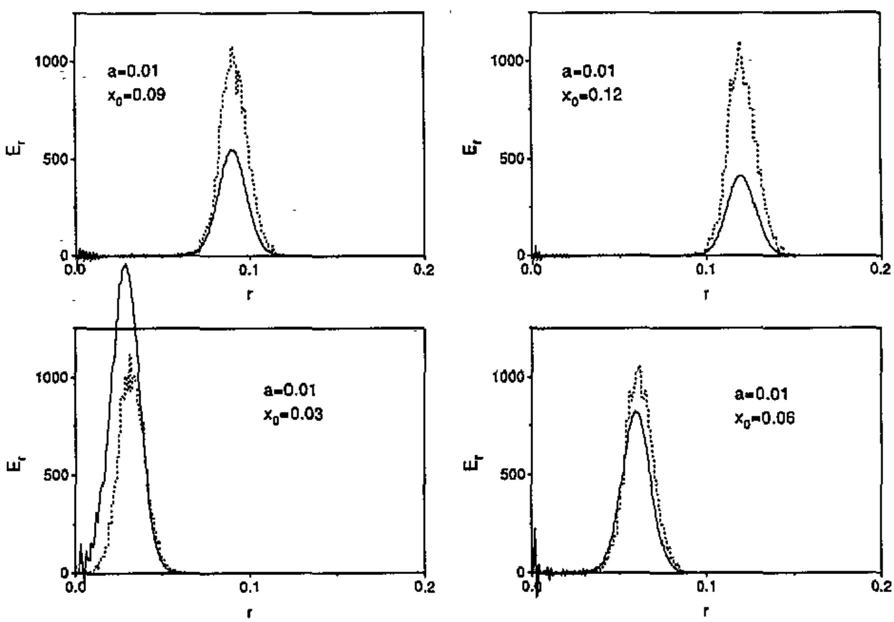

Figure 6. Time evolution of the electric field E_r for a narrow initial probability distribution, with type II initial conditions. The probability distribution (broken curve and not to scale) is shown for comparison.

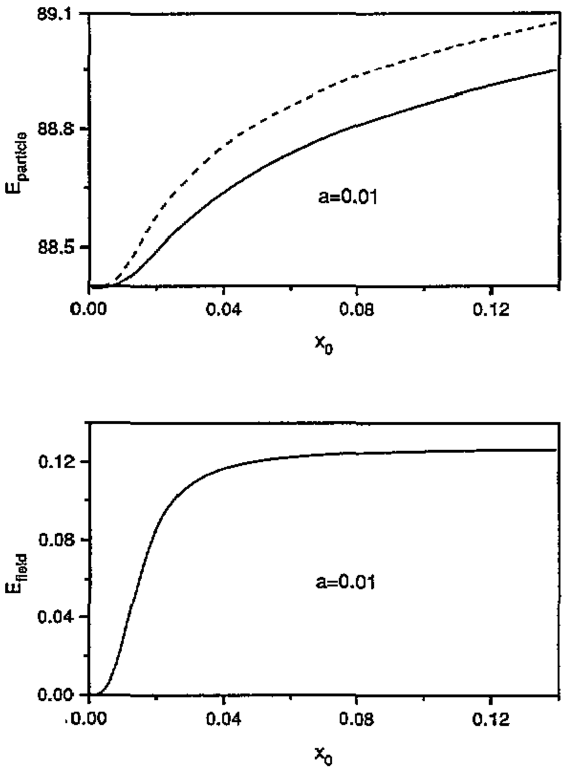

Figure 7. Time dependence of the energy of the field E_{field} and the particle $E_{particle}$ for a narrow initial probability distribution of type II. The broken curve shows the total energy of the system.

energy of the system. Energy conservation no longer holds, because the chosen initial condition does not conserve this quantity: the energy of the field prior to $x_0 = 0$ is zero, and after that it acquires a certain value. The increase in the energy of the particle is not indefinite, and it will eventually stop. However, this happens after the time when part of the electric field overtakes the probability distribution, as will be shown shortly. This may take a long time for a narrow probability distribution. The increase in the particle's energy also indicates that the interaction between the field and particle is strong.

At another extreme are the broad distributions. The time evolution of the electric field for one of these examples is shown in figure 8, where $a = 10$. The electric field (full curve) splits into two parts: one which stays around the probability distribution and the other which travels as a pulse leaving behind the trail of the Coulomb field. This also happens in the case of narrow probability distributions, like the one discussed previously, but the process takes much longer because the two parts of the field stay together for a long time. Eventually the pulse goes to infinity while the rest of the field evolves in time in the same way as the probability distribution with the type I initial condition. The pulse has all the properties of the radiation field, meaning that its total energy is constant (only approximately because the pulse cannot be totally isolated from the rest of the field). The energy of the field, therefore, consists of two parts: one coming from the Coulomb field and the other from the radiation field. The result is that, in the infinite time limit, the total energy of the field reaches a constant value. Its typical time evolution, for the example in figure 8, is shown in figure 9; however, the constant limit at infinite time is not reached.

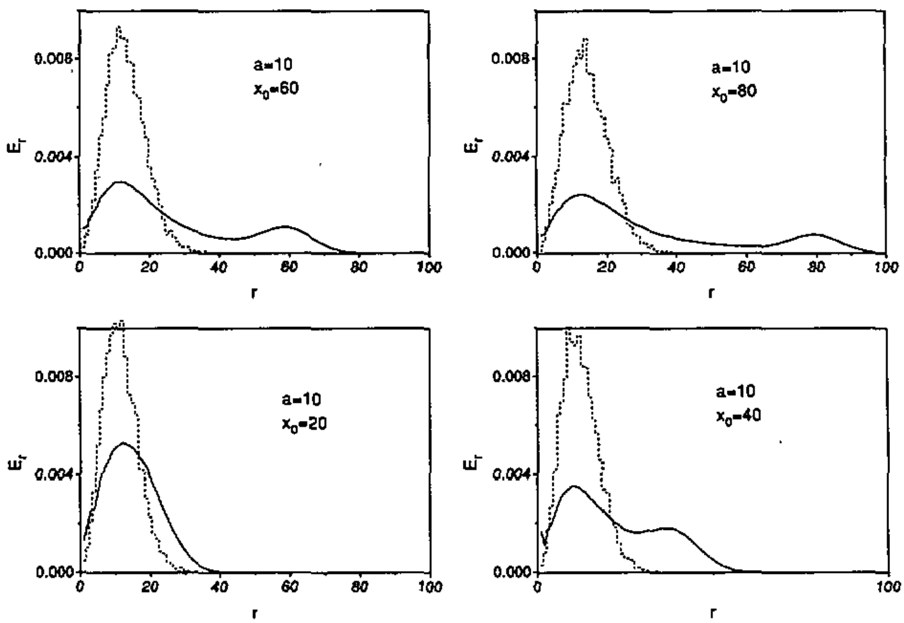

Figure 8. Time evolution of the electric field E_r for a broad initial probability distribution, with type II initial conditions. The probability distribution (broken curve and not to scale) is shown for comparison.

This is the answer to the total energy problem of the previous example. The increase in the total energy, shown in figure 7, continues until the time when the field splits into two distinct parts and, obviously this may take a very long time for very narrow probability distributions.

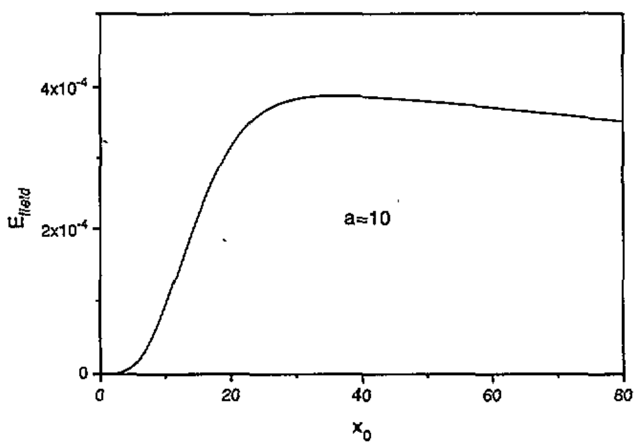

Figure 9. Time dependence of the energy of the field E_{field} for a broad initial probability distribution of type II.

6. Discussion

The pattern of the EM field, which is produced in the self-interaction of a charged particle, was analysed. Only the spherical probability distributions were considered, in which case the magnetic component of the EM field is missing. Two types of initial condition were analysed, each one of them describing a different situation. However, these initial conditions are approximate because some important ingredients are missing. One of these takes into account the presence of another particle, which must inevitably be there if the initial conditions are to be realistic. For example, in type I initial conditions the charged particle is bound prior to $x_0 = 0$ and for that another particle is needed to bind it. Also one needs to know the mechanism by which the particle is set free. On the other hand, in type II initial conditions the probability density is zero prior to $x_0 = 0$ and one needs another particle to satisfy, e.g., the conservation of charge.

Despite these objections the model is not entirely without physical meaning. At least it reveals the dynamics of these probability distributions and, in particular the properties of the EM field. Without any detailed analysis one would expect that, with type I initial conditions, the electric field would very much resemble the Coulomb field of the distribution of charge. The dynamics of the probability distribution determines the pattern of the field, which, for a narrow initial distribution, is the field of a charged spherical strip, with a radius which increases with time. This picture is confirmed in the calculations. However, despite this simple result the radiation reaction force is essential to the dynamics of the particle. Without it the conservation of energy law would be violated. At $x_0 = 0$ the energy of the field has a certain value, whilst at a much later time it is virtually zero. The energy of the field is converted into the energy of particle through the radiation reaction force.

Type II initial conditions are much more interesting. The radiation field is found and its exact interaction with the particle depends on the width of its probability distribution. In all cases, however, the EM field splits into two components: the radiation-type field and the Coulomb-type field. The latter is bound to the probability distribution, whilst the radiation-type field propagates at the speed of light, in the form of a localized pulse, its total energy being constant. The two types of field are indistinguishable for a long time if the initial probability distribution is very narrow. Eventually they separate because they propagate at different velocities and have different amplitudes for large r. The interesting point about this radiation field is that it does not have a magnetic component, and yet it carries momentum. This can be tested by placing a charged particle in this field, in which case the particle would acquire momentum in the radial direction.

In the analysis we have considered spherical probability distributions and, hence, the effect of the radiation reaction force is relatively small but important. The picture may drastically change for particles with spin-$\frac{1}{2}$. It has been shown that the 'structure' of these particles [5] is more complicated than the one which we have considered for spin-0 particles. In the latter one works only with probability distributions in the initial conditions; however, for spin-$\frac{1}{2}$ particles one deals with both the probability density and current. The current circulates round the core of the charge and produces a magnetic dipole which interacts with the motion of the probability density. It is expected that the effect of coupling these two motions may produce much more pronounced dynamics; however, this is the subject of a more elaborate analysis which will be done separately.

In conclusion one can say that the role of the radiation reaction force is important in the dynamics of particles and, beside objections about the initial conditions, there is another one: what is the relevance of these studies when it is known that we are dealing with systems where quantum theory should be applied? Experience shows that classical theory is not far from the true description and, hence, these studies may be considered as a guide to what to expect when quantum theory is applied.

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